

Clinching Auctions Beyond Hard Budget Constraints

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Abstract

Constraints on agent’s ability to pay play a major role in auction design for any setting where the magnitude of financial transactions is sufficiently large. Those constraints have been traditionally modeled in mechanism design as *hard budget*, i.e., mechanism is not allowed to charge agents more than a certain amount. Yet, real auction systems (such as Google AdWords) allow more sophisticated constraints on agents’ ability to pay, such as *average budgets*. In this work, we investigate the design of Pareto optimal and incentive compatible auctions for agents with *constrained quasi-linear utilities*, which captures more realistic models of liquidity constraints that the agents may have. Our result applies to a very general class of allocation constraints known as polymatroidal environments, encompassing many settings of interest such as multi-unit auctions, matching markets, video-on-demand and advertisement systems.

Our design is based Ausubel’s *clinching framework*. Incentive compatibility and feasibility with respect to ability-to-pay constraints are direct consequences of the clinching framework. Pareto-optimality, on the other hand, is considerably more challenging, since the no-trade condition that characterizes it depends not only on whether agents have their budgets exhausted or not, but also on prices at which the goods are allocated. In order to get a handle on those prices, we introduce novel concepts of dropping prices and saturation. These concepts lead to our main structural result which is a characterization of the tight sets in the clinching auction outcome and its relation to dropping prices.

1 Introduction

An important direction in mechanism design is to understand how to design efficient mechanisms when players have constraints on their ability to pay. A first order approximation is to consider *hard budget* constraints, in which each agent has a budget and the mechanism is not allowed to charge him more than this amount. While simpler and more theoretically tractable, hard budgets stand usually as a proxy for more sophisticated payment constraints.

A recent trend in modern internet marketplaces such as Google AdWords is to offer the bidders a better control of their spending by allowing them to express more sophisticated constraints on their ability to pay. A popular feature introduced by Google Adwords in 2010 called “Target CPA bidding” allows advertisers to report *average budget* constraints on top of traditional willingness to pay per item (value) and hard budgets (see [26] for a discussion of this feature in the Google AdWords blog).

It is important to emphasize that values, hard budgets and average budgets play different roles in managing an advertising campaign, or more generally satisfying buyers’ desired goals. In order to illustrate this point, consider a marketplace in which each agent specifies his preferences and gets allocated a certain quantity of a good (ad impressions, for example) and charged a total amount for it. Hard budgets are one of the simplest constraints on the total payment: they specify an upper bound on the total payment. Average budgets specify an upper bound on the ratio of total payment by amount of goods allocated (or alternatively, a lower bound on the ROI, return over investment). On the other hand, individual valuations specify an upper bound on the marginal payment for each individual item, even if some goods are sold at a lower or higher price earlier. To see the difference more clearly, consider an initial outcome where an agent gets some items and pays a certain amount that is below his average budget. If he is offered an extra item for a price less than his value but higher than the average budget, he would prefer the outcome with the extra item as long as the new total payment and allocations don’t exceed his average or hard budget constraints. A natural generalization of average budget constraint is to consider a concave upper bound on the total payment as a function of the number of goods allocated.

We consider here the problem of designing Pareto-optimal mechanisms for settings where the players have *general* (concave) constraints on their ability to pay. This includes hard budgets and average budgets (and combinations thereof), as well as other more sophisticated constraints on the total payment of agents as a function of the set of goods allocated to them. For the special class of hard budgets, a sequence of papers [13, 16, 11, 17, 18, 12] studied this problem for increasingly complex classes of *allocation environments* using Ausubel’s celebrated clinching framework [3] as the main tool. Nonetheless, all those results are restricted to hard budgets, and do not handle more general payment constraints.

Our results and techniques. In this paper, we study the constrained quasi-linear model, in which each agent has a private valuation, and has associated with it a public¹ set of *admissible outcomes* where each outcome is a pair of allocation and payments. The utilities are then quasi-linear if the outcome is admissible and minus infinity otherwise. Our main result is to design an incentive compatible, individually rational and Pareto-efficient auction that handles a general class

¹The assumption that the set of admissible outcomes is public is necessary. Dobzinski et al [13] showed that even for the special case of multi-unit auctions with hard budget constraints, there is no incentive compatible, individually rational and Pareto-optimal auction if budgets are private. Indeed, most papers in the literature on budgets make the public budgets assumption [13, 16, 11, 17, 18, 12], including classical references such as Laffont and Roberts [19] and Maskin [21].

of these payment constraints. It is worth noting that many attempts to generalize clinching auctions to other settings such as private budgets [13], or single-parameter concave valuations [17] have led to impossibility results. In light of that, it is somehow surprising to find an extension for dealing with a general set of payment constraints for which clinching auctions are flexible, and we get a positive result.

Our result applies to a very general class of allocation constraints known as polymatroidal environments. Polymatroidal environments encompass many settings such as multi-unit auctions, advertisement systems, matching markets, video on demand (routing), and spanning tree auctions. See Goel et al [17] and Bikhchandani et al [7] for a more comprehensive discussion on applications of polymatroidal environments.

Algorithmically, our auction can be thought of as a variant of the polyhedral clinching auction in Goel, Mirrokni and Paes Leme [17] with a more general demand function. While applying a variant of the previously known algorithm, proving that the auction is Pareto-optimal for more sophisticated payment constraints becomes considerably more challenging, and requires novel techniques. The reason is as follows: Pareto optimality is usually characterized by a no-trade condition, which states that given two players H and L with values $v_H > v_L$, then for any price p , it should not be possible to take goods away from the L player by paying him at a rate p for the goods taken away and allocating them to the H player charging him a rate p , such that it improves the utility of one of them without making the other worse-off. For hard budgets, there are two obstructions to trade: either H has his budget exhausted in the final solution or the trade violates the allocation feasibility constraints, i.e., H is receiving goods to his maximum capacity. Since neither obstruction depends on the specific price, one can show that if it is possible to trade at price $p \in [v_L, v_H]$, then it is also possible to trade at price $p = v_H$, and vice versa. This implies that one needs to check for no-trade at price $p = v_H$ only, which greatly simplifies the analysis. Now for more general constraints (such as average budget constraint), this is not true. Meaning it might be possible to trade at a price p that is strictly between v_L and v_H but not at price $p = v_H$. The harder part of our analysis is to get a handle on these prices. In order to do so, we define the concept of *dropping prices* which serve as an upper bound on the prices for which trade is possible. We then relate those dropping prices to the feasibility constraints. This leads to our main structural result which is encapsulated in our *Structure of Tight Sets Lemma* (Lemma 4.3 and its Corollary 4.5). We believe that this lemma exposes an interesting structure about clinching auctions that can lead to other applications.

We believe that an important contribution of our work is to show that the clinching framework can be applied to general types of payment constraints. For the special case of hard budgets, clinching has been recently used as a building block to achieve a variety of objectives: Goel et al [18] use it to design online allocation rules, Devanur, Ha and Hartline [12] use it as a building block to approximate revenue in budgeted settings and Dobzinski and Paes Leme [14] use it to approximate an efficiency-related objective. We believe that the ideas in this paper are a first step towards solving other problems (online allocation rules and revenue extraction, for example) for more general types of payment constraints.

Related work Auction design with constraints on player's ability to pay have been extensively studied in the literature. Most of the work is devoted to understand the impact of *hard budget constraints* in standard auctions, see Che and Gale [10] and Benoit and Krishna [5], for example, or optimize the revenue in the presence of budget constraints, as in Laffont and Roberts [19], Borgs et al [8], Chawla et al [9], Malakhov and Vohra [20] and Pai and Vohra [24].

The research line of designing Pareto-optimal incentive-compatible mechanisms with budget constraints was started by Dobzinski, Lavi and Nisan [13], who study agents with hard budget

constraints in a multi-unit auctions setting, i.e., there is a limited supply of identical objects to be sold and the agents have additive valuation over the objects. They also point out that traditional welfare maximization is impossible in budgeted settings and establish Pareto-optimality as the natural efficiency goal for settings with payment constraints. Those ideas were extended in many different directions in subsequent work: Bhattacharya et al [6] study the divisible case and propose budget elicitation schemes, Fiat et al [16] study the same problem for matching markets, Colini-Baldeschi et al [11] study the single-keyword sponsored search setting and Goel et al [17] propose a polyhedral version of this auction that gives a Pareto-optimal auction with hard budgets for any environment that can be modeled as a polymatroid. Such settings include sponsored search, matching markets, and routing auctions. They also prove the impossibility of designing incentive-compatible auctions for very simple polyhedral domains, beyond polymatroidal environments.

For constraints beyond hard budgets, the design of Pareto efficient auctions has been restricted to unit demand settings, where each agent can be allocated at most one item. Aggarwal, Muthukrishnan, Pal and Pal [1], Dutting, Henzinger and Weber [15], Alaei, Jain and Malekian [2], Morimoto and Serizawa [22] use techniques inspired by the Walrasian equilibrium literature to design Pareto-efficient auctions. Baisa [4] studied to which extent efficiency and revenue can be optimized with minimal assumptions on agent's utilities.

2 Setting

2.1 Constrained quasi-linear utilities

We consider a natural generalization of budget constrained utilities which we call *constrained quasi-linear utilities*. In this utility model, player i is characterized by a private valuation v_i and a public *admissible set* $\mathcal{A}_i \subseteq \mathbb{R}_+^2$. Upon getting x_i units of a divisible and homogeneous good and paying π_i dollars for it, we consider that player i 's utility is:

$$u_i(x_i, \pi_i) = \begin{cases} v_i \cdot x_i - \pi_i, & (x_i, \pi_i) \in \mathcal{A}_i \\ -\infty & \text{otherwise} \end{cases}$$

In other words, a player behaves like a quasi-linear player if his outcome (x_i, π_i) is admissible and has minus infinity utility otherwise. For example, budget-constrained utility functions are characterized by $\mathcal{A}_i = \{(x_i, \pi_i) \in \mathbb{R}_+^2; \pi_i \leq B_i\}$. Average budget constraints can be represented by $\mathcal{A}_i = \{(x_i, \pi_i) \in \mathbb{R}_+^2; \pi_i \leq \beta_i x_i\}$. Generally, we consider any admissible set of the form $\mathcal{A}_i = \{(x_i, \pi_i) \in \mathbb{R}_+^2; \pi_i \leq \alpha_i(x_i)\}$ for concave non-decreasing functions *ability-to-pay function* $\alpha_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\alpha_i(0) = 0$.

The condition $\alpha_i(0) = 0$ expresses that a player get zero utility for the zero allocation and zero payments. The fact that α_i is non-decreasing expresses that if a player considers a certain outcome admissible, it also considers admissible any outcome where he is allocated at least as much and pays no more than the original outcome. Finally, the concavity of α_i expresses that if an agent considers certain outcomes admissible, it considers any distribution (convex combination) of such outcomes also admissible.

2.2 Polyhedral Auctions

Given n agents where each agent i has a constrained quasi-linear utility function with value v_i and a valid admissible set \mathcal{A}_i , our main problem is how to auction a divisible good that may be subject to allocation constraints. We assume that the set of allocations is described by a convex set $P \subseteq \mathbb{R}_+^n$

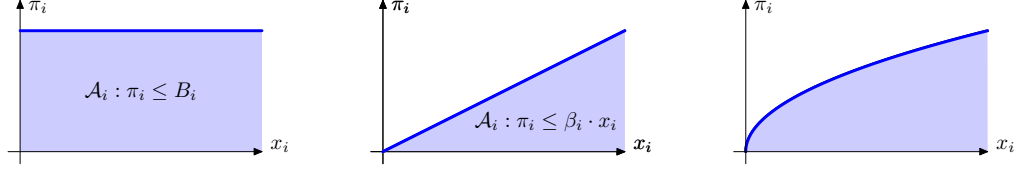


Figure 1: Three examples of admissible regions: the first is an example of *hard budget constraints*, the second of *average budget constraints* and the third is an example of generic *ability-to-pay* function.

such that a point $x = (x_1, \dots, x_n) \in P$ if it is possible to simultaneously allocate x_i units to each player i . We call such a set the *environment*.

We assume that both the admissible sets \mathcal{A}_i and the set of feasible allocations P is public information. The private information of the agents is their value v_i for each unit of the good. An auction is described by two maps: $x : \mathbb{R}_+^n \rightarrow P$ and $\pi : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ such that for each valuation profile $v \in \mathbb{R}_+^n$ it associates an allocation $x(v)$ and payments $\pi(v)$. Our goal is to design an auction that satisfied the following properties:

- *admissibility*, i.e., $(x_i(v), \pi_i(v)) \in \mathcal{A}_i$ for each $i \in [n], v \in \mathbb{R}_+^n$.
- *incentive compatibility* and *individual-rationality*: each player's utility is maximized by reporting their true value regardless of the reports of other agents. Moreover, he always gets non-negative utility by doing so. The classic result by Myerson [23] shows that this is equivalent to $x_i(v_i, v_{-i})$ being monotone in v_i and the payment rule be such that $\pi_i(v_i, v_{-i}) = v_i \cdot x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(u, v_{-i}) du$.
- *Pareto-efficiency*: we say that an outcome is Pareto-efficient if there is no alternative outcome in which all the players' utilities and auctioneer's revenue do not decrease and at least one increases. Formally, an outcome (x, π) is Pareto-efficient if there is no alternative outcome (x', π') with $x' \in P$, $(x'_i, \pi'_i) \in \mathcal{A}_i, \forall i$, $v_i \cdot x'_i - \pi'_i \geq v_i \cdot x_i - \pi_i$, $\sum_i \pi'_i \geq \sum_i \pi_i$ and the sum of those inequalities is strict, i.e., $\sum_i v_i \cdot x'_i > \sum_i v_i \cdot x_i$.

2.3 Polymatroidal Environments

Our results apply to the case where the allocation environment P is a polymatroid. A *polymatroid* is a packing polytope that can be written as:

$$P = \{x \in \mathbb{R}_+^n; \sum_{i \in S} x_i \leq f(S), \forall S \subseteq [n]\}$$

for a monotone submodular function $f : 2^{[n]} \rightarrow \mathbb{R}_+$. A *submodular function* is a set function that has the diminishing marginals property: $f(S \cup i) - f(S) \geq f(T \cup i) - f(T)$ for any subsets $S \subseteq T \subseteq [n]$. An equivalent (and somewhat more traditional) definition is: $f(S \cap T) + f(S \cup T) \leq f(S) + f(T)$ for all $S, T \subseteq [n]$. We say that this function is *monotone* if $f(S) \leq f(T)$ for $S \subseteq T \subseteq [n]$.

Applications of polymatroidal environments are ubiquitous : multi-unit auctions [13], the matching markets [16], sponsored search [17, 11], bandwidth markets, scheduling with deadlines, network planning, video on demand [7], among others. We refer to citegoel12 and [7] for a more extensive discussion on these applications.

2.4 Average Budget Constraints

An important special case of constrained utility functions are *average budget* constraints, in which $\mathcal{A}_i = \{(x_i, \pi_i) \in \mathbb{R}_+^2; \pi_i \leq \beta_i \cdot x_i\}$, where the parameter β_i is called the average budget. In this particular case, the utility function enforces a constraint that the player must pay at most β_i per unit.

At a first glance, this might seem equivalent to a player being quasi-linear with value $\tilde{v}_i = \min\{v_i, \beta_i\}$. In order to see the difference, consider two different agents: (1) agent one has value $v_1 = 2$ and average budget $\beta_1 = 1$ and (2) agent two is quasi-linear with value $v_2 = \min\{v_1, \beta_1\} = 1$. Now, consider the outcome with $x_i = 1$ and $\pi_i = 0$. At this point, consider offering an additional item for each player at the price of 2. The first player would gladly take it, since it is below his value and doesn't violate the average budget constraints, but the second player wouldn't.

Despite of that, both settings are not completely dissimilar. Consider the problem of designing an incentive compatible, individually rational and Pareto-efficient auction to sell one single good to players with average budgets. It is simple to see that running the Vickrey auction on $\tilde{v}_i = \min\{v_i, \beta_i\}$ does the trick. Or, more generally:

Lemma 2.1 *For multi-unit auctions, i.e., $P = \{x \in \mathbb{R}_+^n; \sum_i x_i \leq s\}$, the VCG auction on $\tilde{v}_i = \min\{v_i, \beta_i\}$ is incentive compatible, individually rational and Pareto-efficient auction.*

The proof is trivial and is included in the appendix for completeness. The strategy above, however, doesn't generalize beyond multi-unit auctions. Running VCG on \tilde{v}_i for more general polymatroidal environments is still incentive compatible and individually rational, but fails to be Pareto-efficient in general. In fact, consider the following very simple example:

Example 2.2 *Consider the environment $P = \{x \in \mathbb{R}_+^2; x_1 + x_2 \leq 3, x_1 \leq 2, x_2 \leq 2\}$. For readers familiar with sponsored search, this corresponds to the sponsored search environment with click-through-rates (2,1). Now, consider two agents with average budget constraints and $v_1 = 10, \beta_1 = 1$ and $v_2 = 2, \beta_2 = 2$. Running VCG on \tilde{v} we get allocation $x = (1, 2)$ and $\pi = (0, 1)$. This outcome is clearly not Pareto-efficient, since for any $0 < z \leq \frac{1}{2}$, the outcome $x = (1+z, 2-z)$, $\pi = (2z, 1-2z)$ is a Pareto-improvement (x, π) .*

3 Warm-up: the multi-units environment

As a warm-up we consider the problem of designing an incentive compatible, individually rational and Pareto optimal auction for the multi-units setting, i.e., $P = \{x \in \mathbb{R}_+^n; \sum_i x_i \leq 1\}$ when agents have constrained quasi-linear utilities. This will allow us to highlight the main features of our design in a combinatorially simple setting. The auction we will describe is a discrete step ascending clock price auction, based on the clinching framework. The auction takes as input the value v_i of each agent and their admissible sets \mathcal{A}_i (defined in terms of an ability-to-pay function α_i) and produces a final allocation x_i and payments π_i for each agent. We will denote $\beta_i = \lim_{x \downarrow 0} \alpha_i(x)/x$.

The auction is initialized with zero allocation and zero payments for all the agents $x_i = \pi_i = 0$. The price clock is represented by a vector $p \in \mathbb{R}_+^n$ where p_i represents the price faced by agent i . Prices are initialized to zero.

In round-robin fashion an agent \hat{i} is chosen and his price $p_{\hat{i}}$ is incremented by a fixed amount $\epsilon > 0$. At this point, we compute the *demand* of each agent, which is the maximum amount of the good that this agent would want to acquire at price p_i , i.e., $d_i = \arg\max_z u_i(x_i + z, \pi_i + p_i z)$. It can be computed as follows: $d_i = \max\{z; (x_i + z, \pi_i + p_i z) \in \mathcal{A}_i\}$ if $p_i < v_i$ and $d_i = 0$ otherwise. Based on the demands of each agent, we calculate how much each agent *clinches* in each round,

i.e., how much it is safe to give to each agent while not making any allocation infeasible for other agents.

The clinched amount is calculated as follows: let $s = 1 - \sum_i x_i$ be the remnant supply. The total demand of all agents except i is given by $\sum_{j \neq i} d_j$. We define the difference $\delta_i = [s - \sum_{j \neq i} d_j]^+$ as the clinched amount, i.e., the portion of the remnant supply that is under-demanded by agents $[n] \setminus \{i\}$. The auction proceeds by updating the allocation and payments by giving to each agent his clinched amount at the current price. We summarize the auction in Algorithm 1.

ALGORITHM 1: Multi-Units Clinching Auction

Input: P, v_i, \mathcal{A}_i

$p_i = 0, x_i = 0$, for all i and $\hat{i} = 1$

do

$d_i = \max\{z_i; (x_i + z_i, \pi_i + p_i z_i) \in \mathcal{A}_i\}$ if $p_i < v_i$ and $d_i = 0$ otherwise.

$\delta_i = [1 - \sum_j x_j - \sum_{j \neq i} d_j]^+$,

$x_i = x_i + \delta_i, \quad \pi_i = \pi_i + p_i \cdot \delta_i,$

$d_i = \max\{z_i; (x_i + z_i, \pi_i + p_i z_i) \in \mathcal{A}_i\}$ if $p_i < v_i$ and $d_i = 0$ otherwise.

$p_{\hat{i}} = p_{\hat{i}} + \epsilon, \quad \hat{i} = \hat{i} + 1 \mod n$

while $d \neq 0$

The outcome of the auction corresponds to the final allocation and payments of the ascending procedure. It follows from standard arguments about the clinching framework [3, 13] that this auction is incentive compatible and individually rational. It is individually rational since it never allocates any amount at a rate larger than the value v_i , so in the end, $\pi_i \leq v_i x_i$. It is incentive compatible, since the value only determines when an agent drops his demand to zero. By misreporting his value, an agent can either drop out of the ascending procedure earlier (potentially missing items he could acquire at a price smaller than his valuation) or drop later (potentially acquiring items for price larger than his value). Therefore it is a dominant strategy for each agent to truthfully report his value.

We are left to argue that the auction is Pareto optimal. In order to do that, first, we define the notion of *dropping price* and then we give a structural characterization of the outcome in terms of dropping prices. The main result in this paper (Lemma 4.3) is a generalization of this characterization for generic polymatroidal environments.

In the rest of the section, we will use the following assumption that holds wlog in the limit when ϵ goes to zero:

Assumption 3.1 *All values of v_i and $\beta_i = \lim_{x \downarrow 0} \alpha_i(x)/x$ are multiples of ϵ , which is the price clock increment in the ascending auction.*

3.1 Dropping Prices

Definition 3.2 (Dropping price) *Given an execution of Algorithm 1, we define the dropping price for agent i (we call ϕ_i) as the first price for which he had zero demand.*

The demand of an agent can drop from positive to zero in the execution of Algorithm 1 for three different reasons:

1. the first case is where the buyer clinched his entire demand, i.e., $\delta_i = d_i$. By the definition of demand, the player ends up with an allocation such that $\pi_i = \alpha(x_i)$, since just before clinching, his demand was: $d_i = \max\{z_i; (\tilde{x}_i + z_i, \tilde{\pi}_i + p \cdot z_i) \in \mathcal{A}_i\}$, where $(\tilde{x}_i, \tilde{\pi}_i)$ is his allocation and payment just before clinching for the last time.

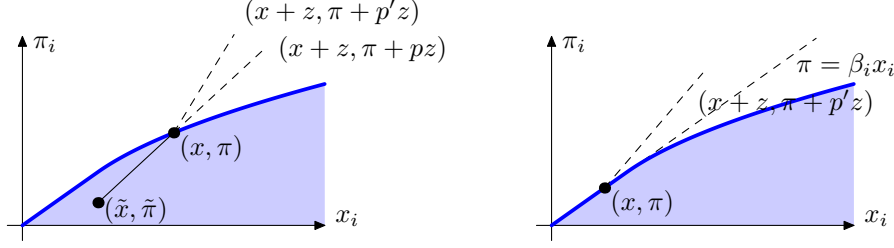


Figure 2: Depiction of two reasons a player might drop his demand to zero. Left: by clinching his entire demand. Right: by having $\pi_i = \beta_i x_i$ and $p > \beta_i$.

After this happens, for any price $p' \geq p$, the demand is zero, since a positive demand would imply that there is some $\kappa > 0$ such that: $(\tilde{x}_i + d_i + \kappa, \tilde{\pi}_i + p \cdot d_i + p' \cdot \kappa) \in \mathcal{A}_i$. This would contradict the maximality of z_i , since by the concavity of α_i , we would have: $(\tilde{x}_i + (d_i + \kappa), \tilde{\pi}_i + p \cdot (d_i + \kappa)) \in \mathcal{A}_i$. This is depicted in the first part of Figure 2.

2. the player didn't clinch his entire demand, but the price reached his value, i.e., $p = v_i$.
3. the player didn't clinch his entire demand, but $\pi_i = \beta_i \cdot x_i$ and $p > \beta_i$. This is depicted in the second part of Figure 2.

We observe that:

Lemma 3.3 *The dropping price ϕ_i is at most v_i . Also, if the final outcome of agent i is $x_i = \pi_i = 0$, then $\phi_i = \min\{\beta_i + \epsilon, v_i\}$.*

Proof : The fact that $\phi_i \leq v_i$ comes from the fact that v_i , p_i and β_i are multiples of ϵ and that for $p_i \geq v_i$, the demand of i is zero. Also, if $x_i = \pi_i = 0$, there are two reasons for the demand to become zero: either p_i becomes larger than β_i or p_i reaches v_i . ■

3.2 Multi-units version of the Structure of Tight Sets Lemma

Now we relate the dropping prices to the structure of the final outcome: we show that if one sorts agents by dropping price, the agents with high dropping price (which we will call H) will be allocated and charged the maximum admissible amount for the quantity they get. Agents with low dropping price (which we will call L) will be unallocated and will be charged zero.

Lemma 3.4 *Let (x, π) be the outcome of the clinching auction for the multi-units setting, then one can partition the set of agents $[n] = L \cup \{k\} \cup H$ such that:*

- for $i \in L$, $x_i = \pi_i = 0$ and $\phi_i \leq \phi_k$.
- for $i \in H$, $v_i > \phi_i \in \{\phi_k, \phi_k - \epsilon\}$ and $\pi_i = \alpha_i(x_i)$.
- k drops without clinching his demand, therefore either (i) $\phi_k = v_k$ or (ii) $\pi_k = \beta_k x_k$ and $\phi_k = \beta_k + \epsilon$.

Moreover, the clinching auction allocates all the goods, i.e., $\sum_i x_i = 1$.

The lemma above is a special case of Lemma 4.3, so we defer a formal proof until that point. The proof of the special case is implicit in Bhattacharya et al [6] and Goel et al [18]. The main idea behind it is to show that all agents that acquire a positive amount drop their demand at essentially the same price: once a player that already acquired a positive amount drops his demand to zero, all other players clinch their entire demand.

3.3 Pareto optimality

Theorem 3.5 *The clinching auction with constrained quasi-linear utilities for the multi-unit setting is Pareto optimal.*

Proof: Let (x, π) be the outcome of Algorithm 1 for valuations v_i and admissible sets \mathcal{A}_i defined by α_i . Assume that an alternative outcome (x', π') is a Pareto improvement, i.e., $u_i(x'_i, \pi'_i) \geq u_i(x_i, \pi_i)$, $\sum_i \pi'_i \geq \sum_i \pi_i$ and $\sum_i v_i x'_i > \sum_i v_i x_i$.

Let L , H and k be as in Lemma 3.4. First we show that $\pi_i - \pi'_i \geq \phi_k(x_i - x'_i)$ for all i and for the case where $x_i < x'_i$, the inequality holds strictly, i.e., $\pi_i - \pi'_i > \phi_k(x_i - x'_i)$. Consider the following cases:

- $i \in L$, then $\pi'_i \leq \min\{\beta_i, v_i\} \cdot x'_i \leq \phi_i x'_i \leq \phi_k x'_i$, where the first inequality follows from the fact that $u_i(x'_i, \pi'_i) \geq 0$, the second follows from Lemma 3.3 and the third from Lemma 3.4. Noting that for $i \in L$, $x_i = \pi_i = 0$, we get: $\pi_i - \pi'_i \geq \phi_k(x_i - x'_i)$
- $i \in H$, $x_i \geq x'_i$: since $v_i > \phi_i \geq \phi_k - \epsilon$ and all values and prices are multiples of ϵ , then: $v_i \geq \phi_k$. Since $u_i(x'_i, \pi'_i) \geq u_i(x_i, \pi_i)$, then $\pi_i - \pi'_i \geq v_i \cdot (x_i - x'_i) \geq \phi_k \cdot (x_i - x'_i)$.
- $i \in H$, $x_i < x'_i$: player i clinched his entire demand at price ϕ_i . By the definition of demand for any $\kappa > 0$, $(x_i + \kappa, \pi_i + \phi_i \cdot \kappa) \notin \mathcal{A}_i$. In particular, for $\kappa = x'_i - x_i$. Since $(x'_i, \pi'_i) \in \mathcal{A}_i$, it must be the case that: $\pi'_i < \pi_i + \phi_i \cdot (x'_i - x_i)$. Now using the fact that $\phi_i \leq \phi_k$ and re-arranging the inequality we get $\pi_i - \pi'_i > \phi_k(x_i - x'_i)$.
- $i = k$: then either (i) $\phi_k = v_k$, in which case we use the fact that (x', π') is a Pareto improvement to get that: $\pi_k - \pi'_k \geq v_k(x_k - x'_k) = \phi_k(x_k - x'_k)$; or (ii) $\pi_k = \beta_k x_k$ and $\phi_k = \beta_k + \epsilon$. If $x_k \geq x'_k$, we use the same argument as in the second item. If $x_k < x'_k$, we use that (x'_k, π'_k) is admissible therefore $\pi'_k \leq \beta_k x'_k$ so: $\pi_k - \pi'_k \geq \beta_k(x_k - x'_k) > \phi_k(x_k - x'_k)$.

Summing for all i we obtain $\sum_i \pi_i - \sum_i \pi'_i \geq \phi_k \cdot (\sum_i x_i - \sum_i x'_i) = \phi_k(1 - \sum_i x'_i) \geq 0$. Since $\sum_i \pi_i \leq \sum_i \pi'_i$, the revenue in both cases must be equal, therefore all inequalities must hold with equalities. In particular, it must be that $x_i \geq x'_i$ for all i since for $x_i < x'_i$, the inequality $\pi_i - \pi'_i \geq \phi_k(x_i - x'_i)$ holds strictly. This in particular implies that $\sum_i v_i x_i \geq \sum_i v_i x'_i$, contradicting the fact that (x', π') is a Pareto improvement. ■

4 Polymatroidal environments

Now we extend the result in the previous section to general polymatroidal environments. We do so by changing the way demands are calculated in the polyhedral clinching auction of Goel et al [17]. As usual, incentive compatibility and individual rationality follow as usual from properties of the clinching framework. The main challenge in extending the result in the previous section to general polymatroidal environments is extending Lemma 3.4 to combinatorial settings.

We begin by describing the polyhedral clinching auction for the case of constrained quasi-linear utilities. The auction takes as input the feasible set $P \subseteq \mathbb{R}_+^n$, agent values v_i and valid admissible sets \mathcal{A}_i and computes an allocation $x \in P$ and a payment vector π such that $(x_i, \pi_i) \in \mathcal{A}_i$ for all i .

The auction, described in Algorithm 2, is a version of Algorithm 1 that redefines the clinching step to take into account the environment P .

ALGORITHM 2: Polyhedral Clinching Auction

Input: P, v_i, \mathcal{A}_i

$p_i = 0, x_i = 0$, for all i and $\hat{i} = 1$

do

$d_i = \max\{z_i; (x_i + z_i, \pi_i + p_i z_i) \in \mathcal{A}_i\}$ if $p_i < v_i$ and $d_i = 0$ otherwise.

$\delta = \text{clinch}(P, x, d)$,

$x_i = x_i + \delta_i, \quad \pi_i = \pi_i + p_i \cdot \delta_i$,

$d_i = \max\{z_i; (x_i + z_i, \pi_i + p_i z_i) \in \mathcal{A}_i\}$ if $p_i < v_i$ and $d_i = 0$ otherwise.

$p_{\hat{i}} = p_{\hat{i}} + \epsilon, \quad \hat{i} = \hat{i} + 1 \mod n$

\leftarrow point (\clubsuit)

while $d \neq 0$

Definition 4.1 (Clinching) *Given an allocation x and demands d , the remnant supply polytope is defined as $P_{x,d} = \{y \in \mathbb{R}_+^n; x + y \in P; y \leq d\}$. Given an amount z_i for player i we define the polytope on $P_{x,d}^i(z_i)$ of the possible allocations for $[n] \setminus i$ if we allocate extra z_i units to player i . Formally: $P_{x,d}^i(z_i) = \{z_{-i} \in \mathbb{R}_+^{[n] \setminus i}; (z_i, z_{-i}) \in P_{x,d}\}$. Now, the clinching amount δ_i is defined as the maximum allocation to player i that doesn't make any allocation for other players infeasible: $\delta_i = \max\{z_i; P_{x,d}^i(z_i) = P_{x,d}^i(0)\}$.*

It follows from standard arguments on clinching auctions that the auction is incentive-compatible, individually-rational and produces admissible outcomes. See for example Lemmas 3.3, 3.4 and 3.5 in [17].

Theorem 4.2 ([17]) *The clinching procedure is well defined (i.e. it stops after finite time and $x \in P$). The auction produced is truthful, individually rational and produces acceptable outcomes, i.e., $(x_i, \pi_i) \in \mathcal{A}_i$.*

The rest of the paper is dedicated to prove that the auction produces Pareto-efficient outcomes. The proof of Pareto-efficiency is based on a *structural lemma* that relates the tight sets (i.e., sets of agents where $\sum_{i \in S} x_i = f(S)$) to the dropping price as defined in Section 3.1. We note that the definitions and observations in that section are valid for any environment.

Before we get to those, we introduce some new notation. For a vector $x \in \mathbb{R}^n$ and a subset $S \subseteq [n]$ we denote $x(S) := \sum_{i \in S} x_i$. Also, for the remainder of the paper, we focus on a polymatroidal environment P defined by a submodular function f , i.e., $P = \{x \in \mathbb{R}_+^n; x(S) \leq f(S), \forall S \subseteq [n]\}$.

We will also keep the same notation used in the previous section: ϕ_i for dropping prices, β_i for $\lim_{x \downarrow 0} \alpha_i(x)/x$. We will also assume, as before, that v_i and β_i are multiples of ϵ .

4.1 Pareto efficiency via Structure of tight sets

Now we are ready to state the central pieces used to prove Pareto efficiency:

Lemma 4.3 (Structure of tight sets) *In any given iteration of Algorithm 2, after clinching is performed and demands are re-calculated (point (\clubsuit) in the description), if S is the set of players*

with positive demand, then S is tight in the final outcome, i.e., $x(S) = f(S)$ where (x, π) is the final outcome.

Lemma 4.4 *If a player clinches his entire demand in a certain iteration of the auction, then there is a player that in the same iteration drops without clinching his entire demand.*

We defer proving those lemmas until the end of Section 4.3. Before, we discuss how to use this Lemma to prove Pareto-efficiency. Since it is clear that the set of players with positive demand is shrinking, it gives us a natural nested family of tight sets. Moreover, it is a family where we can bound the prices for which they acquire goods during the process:

Corollary 4.5 *Given an execution of 2, let i_1, \dots, i_k be the agents who drop their demand to zero without clinching their entire demand sorted in reverse order in which those events (demand dropping to zero) happens. Also, let $\phi_{i_1} \geq \dots \geq \phi_{i_k}$ be the dropping prices for each agent. Then by taking S_j to be the set of agents with positive demand just before player i_j dropped his demand to zero, we have a nested family of tight subsets:*

$$\emptyset = S_0 \subset S_1 \subset S_2 \subset \dots \subset S_k = [n], \quad x(S_j) = f(S_j), \forall j$$

with the property that: $i_j \in T_j := S_j \setminus S_{j-1}$ and all players $i \in T_j \setminus \{i_j\}$ clinched their entire demand and $\phi_i \in \{\phi_{i_j} - \epsilon, \phi_{i_j}\}$.

Proof : The proof follows directly from Lemma 4.4. The fact that for $i \in T_j$, $\phi_i \in \{\phi_{i_j} - \epsilon, \phi_{i_j}\}$ follows from Lemma 4.3 since in the iteration player i_j drops, the prices of all the agents are either ϕ_{i_j} or $\phi_{i_j} - \epsilon$. ■

The reader familiar with [17] will note the similarity of the nested family in Corollary 4.5 and the nested family in the proof of Lemma 3.8 in [17]. From the proof in [17] it should be clear that getting such a tight family is the main ingredient in proving Pareto-efficiency. The difficulty here is that we need a family which is somehow tied to prices, which wasn't necessary in [17]. There, one could simply use values v_i as proxies prices ϕ_i , since the admissible sets were very simple.

Theorem 4.6 *The outcome of the Polyhedral Clinching Auction (Algorithm 2) is Pareto-efficient.*

Proof : Let (x, π) be the outcome of the clinching auction and assume that there is an alternative outcome (x', π') such that $v_i \cdot x'_i - \pi'_i \geq v_i \cdot x_i - \pi_i, \forall i$, $\sum_i \pi'_i \geq \sum_i \pi_i$ and at least one of those inequalities is strict, which means that the sum of those inequalities is a strict inequality: $\sum_i v_i \cdot x'_i > \sum_i v_i \cdot x_i$. Also, let x' is a feasible point of P , $(x'_i, \pi'_i) \in \mathcal{A}_i, \forall i$.

Assuming the structure in Corollary 4.5, first we show that for $i \in T_j$, $\pi_i - \pi'_i \geq \phi_{i_j}(x_i - x'_i)$. In order to show that, we consider three cases:

- $x_i \geq x'_i, i \neq i_j$: since i clinched his entire demand, then $\phi_i \leq v_i - \epsilon$ and since $\phi_i \in \{\phi_{i_j} - \epsilon, \phi_{i_j}\}$, then: $\phi_{i_j} \leq v_i$. Then $\pi_i - \pi'_i \geq v_i \cdot (x_i - x'_i) \geq \phi_{i_j} \cdot (x_i - x'_i)$.
- $x_i < x'_i, i \neq i_j$: player i clinched his entire demand at price ϕ_i . By the definition of demand for any $\kappa > 0$, $(x_i + \kappa, \pi_i + \phi_i \cdot \kappa) \notin \mathcal{A}_i$. In particular, for $\kappa = x'_i - x_i$. Since $(x'_i, \pi'_i) \in \mathcal{A}_i$, it must be the case that: $\pi'_i < \pi_i + \phi_i \cdot (x'_i - x_i)$. Now using that $\phi_i \leq \phi_{i_j}$ and re-arranging the inequality we get $\pi_i - \pi'_i > \phi_{i_j}(x_i - x'_i)$.

- $i = i_j$: in this case: (i) either $\phi_{i_j} = v_{i_j}$, in which case $\pi_{i_j} - \pi'_{i_j} \geq v_{i_j} \cdot (x_{i_j} - x'_{i_j}) = \phi_{i_j} \cdot (x_{i_j} - x'_{i_j})$, (ii) or $\phi_{i_j} > \beta_{i_j}$ and $\pi_{i_j} = \beta_{i_j} \cdot x_{i_j}$. Since $\pi'_{i_j} \leq \beta_{i_j} \cdot x_{i_j}$, we have that: $\pi_{i_j} - \pi'_{i_j} \geq \beta_{i_j} \cdot (x_{i_j} - x'_{i_j}) > \phi_{i_j} \cdot (x_{i_j} - x'_{i_j})$ if $x_{i_j} \leq x'_{i_j}$. In the case where $x_{i_j} \geq x'_{i_j}$, we can simply use that $\phi_{i_j} \leq v_{i_j}$ and then: $\pi_{i_j} - \pi'_{i_j} \geq v_{i_j} \cdot (x_{i_j} - x'_{i_j}) \geq \phi_{i_j} \cdot (x_{i_j} - x'_{i_j})$.

Now, summing this inequality for all $i \in T_j$, we get:

$$\sum_{i \in T_j} \pi_i - \pi'_i \geq \phi_{i_j} \cdot \sum_{i \in T_j} x_i - x'_i = \phi_{i_j} \cdot (x(T_j) - x'(T_j)) \geq 0$$

since $x(T_j) = f(T_j) \geq x'(T_j)$. Therefore:

$$\sum_{i \in S_j} \pi_i - \pi'_i = \sum_{k \leq j} \sum_{i \in T_k} \pi_i - \pi'_i \geq \phi_{i_j} \cdot (x(S_j) - x'(S_j)) + \sum_{k < j} (\phi_{i_k} - \phi_{i_{k+1}}) \cdot (x(S_k) - x'(S_k)) \geq 0$$

For $j = k$, $S_k = [n]$, so: $0 \geq \sum_{i \in [n]} \pi_i - \pi'_i \geq 0$, where the first inequality comes from Pareto-efficiency and the second inequality comes from the line above. This means that all inequalities along the way used to prove the inequality above should be tight. This means in particular that for all $i \in T_j$, either $x_i = x'_i$ or $v_i = \phi_{i_j}$, since $x_i \neq x'_i$ and $v_i > \phi_{i_j}$ would imply a strict inequality in some of the cases above. Therefore:

$$\sum_{i \in T_j} v_i \cdot (x_i - x'_i) = \sum_{i \in T_j, v_i = \phi_{i_j}} \phi_{i_j} \cdot (x_i - x'_i) = \sum_{i \in T_j} \phi_{i_j} \cdot (x_i - x'_i) \geq \phi_{i_j} \cdot (f(T_j) - x'(T_j)) \geq 0$$

Summing for all j , we get that $\sum_i v_i \cdot x_i \geq \sum_i v_i \cdot x'_i$ contradicting the assumption that $\sum_i v_i \cdot x'_i > \sum_i v_i \cdot x_i$. ■

Relation to the proof for hard budget constraints: In the proof of Lemma 3.8 in [17] for $\mathcal{A}_i = \{(x_i, \pi_i) \in \mathbb{R}_+^2; \pi_i \leq B_i\}$, the case $x_i < x'_i, i \neq i_j$ is simpler, since one can use that $\pi_i = B_i \geq \pi'_i$ together with $v_i x_i < v_i x'_i$ to prove that: $\pi_i - \pi'_i \geq 0 > v_i(x_i - x'_i)$. The entire proof can be done therefore using values v_i instead of dropping prices ϕ_i . Since there is not a global upper bound on payment, this approach does not work for a general admissible set.

4.2 Basic Facts on Polymatroids and Clinching

In the previous subsection we showed that the Pareto-optimality follows from Lemmas 4.4 and 4.3. Proving those statements is the most technically challenging part of the paper. Before we do it, we would like to review some elementary facts about polymatroids. See [25] for example, for an extensive exposition on polymatroids.

Lemma 4.7 (Uncrossing) *If P is the polymatroid defined by $f : 2^{[n]} \rightarrow \mathbb{R}_+$, and $x \in P$ such that for subsets $S, T \subseteq [n]$, $x(S) = f(S)$ and $x(T) = f(T)$ (we say those sets are tight), then $x(S \cap T) = f(S \cap T)$ and $x(S \cup T) = f(S \cup T)$.*

Lemma 4.8 (Polymatroid \cap Box) *If P is the polymatroid defined by $f : 2^{[n]} \rightarrow \mathbb{R}_+$, then $P_{x,d} = \{y \in \mathbb{R}_+^n; x + y \in P; y \leq d\}$ is the polymatroid defined by the (possibly non-monotone) submodular function $\tilde{f}(S) = \min_{T \subseteq S} [f(T) - x(T) + d(S \setminus T)]$.*

Now, we also review a basic fact about clinching, which was proved in [17] :

Lemma 4.9 (Constructive Clinching) *Given current allocation x and payments π and a polymatroid P , the clinched amount δ_i can be calculated as $\delta_i = [\max_{y \in P_{x,d}} \mathbb{1}^t y] - [\max_{y \in P_{x,d}} \mathbb{1}_{-i}^t y_{-i}]$. An alternative description is: $\delta_i = [\tilde{f}([n]) - \tilde{f}([n] \setminus i)]^+$ for the \tilde{f} function defined in the previous lemma.*

We note that the \tilde{f} function define in Lemma 4.8 might not be monotone. The non-monotonicity has to do with the $-x(T)$ term in the definition. It simple to see that if $x = 0$, then $\tilde{f}(\cdot)$ is monotone, since for $S \subseteq S'$: $\tilde{f}(S') = f(T) + d(S' \setminus T) \geq f(T \cap S) + d(S \setminus T) \geq \tilde{f}(S)$, where T is the subset of S' minimizing $f(T) + d(S' \setminus T)$. The following lemma will allow us to define clinching in terms of a monotone submodular function:

Lemma 4.10 *Given a polymatroid P defined by f , $x \in P$ and a demand vector $d \in \mathbb{R}_+^n$, then:*

$$\max_{y \in P_{x,d}} \mathbb{1}^t y = \max_{y \in P_{0,x+d}} \mathbb{1}^t y - \mathbb{1}^t x$$

Proof : The problem $\max_{y \in P_{x,d}} \mathbb{1}^t y$ can be written as $\max \mathbb{1}^t y$ s.t. $(x + y)(S) \leq f(S); \forall S; 0 \leq y \leq d$. Once we relax $0 \leq y$ to $-x \leq y$ and substitute $z = x + y$ we get the problem: $\max \mathbb{1}^t (z - x)$ s.t. $z(S) \leq f(S); \forall S; 0 \leq z \leq x + d$. So, it is simple to see that $\max_{y \in P_{0,x+d}} \mathbb{1}^t y - \mathbb{1}^t x$ is a relaxation of the first problem. Now, among the *optimal* solutions z to the first problem, take one minimizing $\Phi(z) = \sum_i (x_i - z_i)^+$, i.e., take the optimal solution that minimally violates $x \leq z$. We wish to prove that $\Phi(z) = 0$.

Assume by contradiction that $\Phi(z) > 0$. If this is true, then $S^- = \{i; z_i < x_i\} \neq \emptyset$. Since $\mathbb{1}^t z \geq \mathbb{1}^t x$ (after all $z = x$ is feasible), then $S^+ = \{i; z_i > x_i\} \neq \emptyset$ as well. Now, notice that for all $i \in S^-$, $j \in S^+$ and $\delta > 0$, the solution $(z_i + \delta, z_j - \delta, z_{-i,j})$ can't be feasible, otherwise we would violate the minimality of Φ . Therefore, there must be a tight set T_{ij} between i and j , i.e., $z(T_{ij}) = f(T_{ij})$, $i \in T_{ij}$, $j \notin T_{ij}$. Using uncrossing (Lemma 4.7), the set $T = \cup_{i \in S^-} \cap_{j \in S^+} T_{ij}$ we get a set that is tight, i.e., $z(T) = f(T)$, $S^- \subseteq T$ and $T \cap S^+ = \emptyset$. So, $f(T) = z(T) < x(T)$, where the second inequality comes from the fact that $S^- \subseteq T \subseteq [n] \setminus S^+$. This contradicts the fact that $x(T) \leq f(T)$ because $x \in P$. ■

The previous lemma motivates the following notation for submodular functions capped by a vector: given a vector $\psi \in \mathbb{R}_+^n$, define $f_\psi(S) = \min_{T \subseteq S} f(T) + \psi(S \setminus T)$, which is the submodular function defining $P_{0,\psi}$. Using this new notation together with the previous lemmas, we get:

Corollary 4.11 *The clinched amount can be calculated as: $\delta_i = f_{x+d}([n]) - f_{x+(0,d_{-i})}([n])$.*

Proof : Notice that $\max_{y \in P_{x,d}} \mathbb{1}_{-i}^t y_{-i} = \max_{y \in P_{x,(0,d_{-i})}} \mathbb{1}^t y = f_{x+(0,d_{-i})}([n]) - \mathbb{1}^t x$ ■

4.3 Saturation and a proof of the Structure of Tight Sets Lemma

This sets the stage to the definition of *saturation*, which will be fundamental concept in the following proofs. First we give an intuitive notion of saturation and then we give a more practical equivalent definition using the f_ψ notation.

Definition 4.12 (Saturation) *Given x, d in a certain point of the execution of the Polyhedral Clinching Auction (Algorithm 2) we say that agent i is saturated if there is an optimal solution to $\max_{z \in P_{0,x+d}} \mathbb{1}^t z$ with $z_i < x_i + d_i$. We say that i is unsaturated if all optimal solutions are such that $z_i = x_i + d_i$.*

Also, for a fixed agent k , we say that agent i is k -saturated if, for ψ such that $\psi_{-k} = x_{-k} + d_{-k}$ and $\psi_k = x_k$, there are optimal solutions to $\max_{z \in P_{0,\psi}} \mathbb{1}^t z$ with $z_i < \psi_i$. We say that i is k -unsaturated if all optimal solution are such that $z_i = \psi_i$.

Intuition of the concept of saturation and a connection to previous work: For the special case of multi-unit auctions $P = \{x; \sum_i x_i \leq 1\}$ studied in [13, 6, 18], an important concept is that of the clinching set – which is the set of players who clinch some amount of the good as the price increases. For generic polymatroidal setting, the concept of saturation emulates the concept of the clinching set in the following sense: player i will be able to clinch as the demand of k drops iff k is i -unsaturated. For multi-unit auctions, the set of k -unsaturated elements are either \emptyset or $[n]$. So, one can represent this structure by defining the *clinching set* as the set of players k for which all $[n]$ are k -unsaturated.

The next lemma gives a more practical definition of saturation:

Lemma 4.13 *Let ψ be such that $\psi_{-k} = x_{-k} + d_{-k}$ and $\psi_k = x_k$. Then agent i is k -unsaturated iff $\psi_i = f_\psi([n]) - f_\psi([n] \setminus i)$.*

Proof : The optimal solution of $\max_{z \in P_{0,\psi}} \mathbb{1}^t z$ that minimizes z_i is the one that maximizes $\mathbb{1}_{-i}^t z_{-i}$. Since the feasible set is a polymatroid, one can simply greedily increase each coordinate as much as one can, leaving i as the last one. Therefore we get $z_i = f_\psi([n]) - f_\psi([n] \setminus i)$. Now, if $z_i = \psi_i$, then i is k -unsaturated, if $z_i < \psi_i$, then i is k -saturated. ■

Now, we state and prove two useful lemmas on dealing with submodular functions capped by a vector:

Lemma 4.14 *Given ψ and $\psi' = (\psi'_i, \psi_{-i})$ with $\psi'_i < \psi_i$ and $i \in S$, the following identity holds: $f_{\psi'}(S) = \min\{f_\psi(S), f_\psi(S \setminus i) + \psi'_i\}$.*

Proof : By the definition of $f_{\psi'}$, there is a set $T \subseteq S$ such that $f_{\psi'}(S) = f(T) + \psi'(S \setminus T)$. If $i \in T$, then $f_{\psi'}(S) = f_\psi(S)$. If not, then: $f_{\psi'}(S) = \psi'_i + f(T) + \psi'(S \setminus T, i) = \psi'_i + f(T) + \psi(S \setminus T, i) = \psi'_i + f_\psi(S \setminus i)$ by the fact that T minimizes $f(T) + \psi(S \setminus T, i)$ for $T \subseteq S \setminus i$. ■

Lemma 4.15 *Given an allocation x and demands d in a certain point of the algorithm, let U be the set of k -unsaturated elements and S be the set of k -saturated elements. If $\psi = (x_k, x_{-k} + d_{-k}) \in \mathbb{R}_+^n$ and $S \subseteq X \subseteq [n]$, then: $f_\psi(X) = f(S) + \psi(X \cap U)$.*

Proof : By the definition of unsaturated elements, $i \in U$ if $\psi_i = f_\psi([n]) - f_\psi([n] \setminus i)$. By submodularity, for every $i \in X \subseteq [n]$, $\psi_i \geq f_\psi(X) - f_\psi(X \setminus i) \geq \psi_i$, so: $f_\psi(X) - f_\psi(X \setminus i) = \psi_i$. So: Let i_1, \dots, i_k be the elements in $X \cap U$, so:

$$\begin{aligned} f_\psi(X) &= f_\psi(S) + \sum_{j=1}^k f_\psi(S \cup \{i_1, \dots, i_j\}) - f_\psi(S \cup \{i_1, \dots, i_{j-1}\}) = \\ &= f_\psi(S) + \sum_{j=1}^k \psi_{i_j} = f_\psi(S) + \psi(X \cap U) \end{aligned}$$

Now we need to show that $f_\psi(S) = f(S)$. In fact, by the definition of f_ψ , $f_\psi(S) = f(T) + \psi(S \setminus T)$. If $S \setminus T \neq \emptyset$, then take $i \in S \setminus T$, so:

$$\psi_i \geq f_\psi(S) - f_\psi(S \setminus i) = f(T) + \psi(S \setminus T) - f_\psi(S \setminus i) \geq f(T) + \psi(S \setminus T) - [f(T) + \psi(S \setminus (T \cup i))] = \psi_i$$

Using the identity we just proved, we have:

$$\psi_i \geq f_\psi([n]) - f_\psi([n] \setminus i) \geq [f_\psi(S) + \psi([n] \setminus S)] - [f_\psi(S \setminus i) + \psi([n] \setminus S)] = \psi_i$$

which leads to the conclusion that i is k -unsaturated, contradicting the definition of S . Therefore it must be the case that $S \setminus T = \emptyset$, i.e., $f_\psi(S) = f(S)$. \blacksquare

Now, we state a sequence of three invariants that are maintained throughout the auction. The first two of them are proved in [17], while the third one has to do with the concept of saturation introduced in this paper. The proofs can be found in Appendix A.

Lemma 4.16 (Invariant I: Maximality of clinching) *Performing the clinching step twice in a row without updating prices will result in no amount clinched in the second step. Alternatively: in point (\clubsuit) of the execution of the algorithm: $f_{x+d}([n]) = f_{x+(0,d_{-i})}([n])$ for all $i \in [n]$.*

Lemma 4.17 (Invariant II: All goods sold) *At any point of the execution, it is always possible to sell all the goods. Alternatively: $f_{x+d}([n]) = f([n])$.*

Lemma 4.18 (Invariant III: Self-unsaturation) *Through the execution of the clinching auction, player i is i -unsaturated for every i . Alternatively: $f_{x+(0,d_{-i})}(S) = f_{x+(0,d_{-i})}(S \setminus i) + x_i$ for all $i \in [n]$ and $S \ni i$.*

Now, we are ready to prove Lemma 4.4:

Proof of Lemma 4.4 : Let x, d be the allocation and demands of players at point (\clubsuit) of the execution of Algorithm 2. If before the time the algorithm is in point (\clubsuit) some player drops his demand to zero, we will show that either this player is \hat{i} (the player for which the price increased) or $d_{\hat{i}}$ became zero.

If $d_{\hat{i}}$ was zero already, nothing changes when the price $p_{\hat{i}}$ increases, so no clinching happens. On the other hand, if after $p_{\hat{i}}$ increases and demands are updated, the new demand of \hat{i} (let's call it $d'_{\hat{i}}$) is still positive, we will see that $\delta_i < d_i$ for all players with $i \neq \hat{i}$ and $d_i > 0$.

Let's call $d' = (d'_{\hat{i}}, d_{-\hat{i}})$. So, $\delta_i = f_{x+d'}([n]) - f_{x+(0,d'_{-i})}([n])$. First, notice that: $f_{x+d'}([n]) \geq f_{x+(0,d_{-\hat{i}})}([n]) = f_{x+d}([n]) \geq f_{x+d'}([n])$, where the inequalities come from $d' \geq (0, d_{-\hat{i}})$ and $d \geq d'$. The equality comes from Lemma 4.16. Therefore: $f_{x+d'}([n]) = f_{x+d}([n])$.

Also, by Lemma 4.14, $f_{x+(0,d'_{-i})}([n]) = \min\{f_{x+(0,d_{-i})}([n]), f_{x+(0,d_{-i})}([n] \setminus \hat{i}) + x_{\hat{i}} + d'_{\hat{i}}\}$. If $f_{x+(0,d'_{-i})}([n]) = f_{x+(0,d_{-i})}([n])$, then: $\delta_i = f_{x+d}([n]) - f_{x+(0,d_{-i})}([n]) = 0 < d_i$ by Lemma 4.16. If $f_{x+(0,d'_{-i})}([n]) = f_{x+(0,d_{-i})}([n] \setminus \hat{i}) + x_{\hat{i}} + d'_{\hat{i}}$, then:

$$\begin{aligned} \delta_i &= f_{x+d}([n]) - [f_{x+(0,d_{-i})}([n] \setminus \hat{i}) + x_{\hat{i}} + d'_{\hat{i}}] \stackrel{4.18}{=} f_{x+d}([n]) - [f_{x+(0,d_{-i})}([n] \setminus \hat{i}, i) + x_i + x_{\hat{i}} + d'_{\hat{i}}] \stackrel{4.16}{=} \\ &= f_{x+(0,d_{-i})}([n]) - [f_{x+(0,d_{-i})}([n] \setminus \hat{i}, i) + x_i + x_{\hat{i}} + d'_{\hat{i}}] = \\ &= f_{x+(0,d_{-i})}([n]) - f_{x+(0,d_{-i})}([n] \setminus \hat{i}) + f_{x+(0,d_{-i})}([n] \setminus \hat{i}) - f_{x+(0,d_{-i})}([n] \setminus \hat{i}, i) - [x_i + x_{\hat{i}} + d'_{\hat{i}}] \stackrel{4.18}{\leq} \\ &\leq x_{\hat{i}} + x_i + d_i - [x_i + x_{\hat{i}} + d'_{\hat{i}}] = d_i - d'_{\hat{i}} < d_i \end{aligned}$$

\blacksquare

Proof of Lemma 4.3 : Let x, d be the allocation and demands during the execution of the algorithm in point (\clubsuit) . We will show if $T = \{i; d_i > 0\}$, then $x([n] \setminus T) = f([n]) - f(T)$. If we show that, then we are done, since for the players in $[n] \setminus T$, the value of x coincides with the final

allocation, so, if x^f is the final allocation, we know that $x^f([n] \setminus T) = x([n] \setminus T)$. By Lemma 4.17, $x^f([n]) = f([n])$, so: $x^f(T) = f(T)$.

In the first iteration $T = [n]$, so the property $x([n] \setminus T) = f([n]) - f(T)$ holds trivially. Now we only need to show that this property is preserved from one iteration to another. Let x, d be the allocation and demands when the algorithm is at point (\clubsuit) . Also, let \hat{i} be the player whose price increases just after point (\clubsuit) . If $d_{\hat{i}}$ doesn't drop to zero, no other player's demand drops to zero (Lemma 4.4), so T remains unchanged and the property is preserved.

If on the other hand, $d_{\hat{i}}$ becomes zero, we claim that the players whose demand drops to zero are exactly the \hat{i} -unsaturated players. Letting $d' = (0, d_{-\hat{i}})$ and then using Lemma 4.14 we can see that:

$$\begin{aligned} \delta_i &= f_{x+d'}([n]) - f_{x+(0,d'_i)}([n]) = f_{x+d'}([n]) - \min\{f_{x+d'}([n]), f_{x+d'}([n] \setminus i) + x_i\} \\ &= [f_{x+d'}([n]) - f_{x+d'}([n] \setminus i) - x_i]^+ \end{aligned}$$

Therefore $\delta_i = d_i$ iff $f_{x+d'}([n]) - f_{x+d'}([n] \setminus i) = x_i + d_i$, i.e., i is \hat{i} -unsaturated. Let U and S be respectively the sets of \hat{i} -unsaturated in T and \hat{i} -saturated players in T . Notice that the players in U will be ones who will join the set of players with zero demand in the next iteration and their final allocation (let's call it x^f) will be $x_i^f = x_i + d_i$ for $i \in U \setminus \hat{i}$ and $x_i^f = x_i$. The set S will be the players who will have non-zero demand the next time we reach point (\clubsuit) .

In order to use Lemma 4.15, we note that the players outside T are \hat{i} -unsaturated, since for $k \notin T$, $x + (0, d_{-\hat{i}}) \leq x + d = x + (0, d_{-k})$, therefore: $f_{x+(0,d_{-\hat{i}})}([n] \setminus k) \leq f_{x+(0,d_{-k})}([n] \setminus k) = f_{x+(0,d_{-k})}([n]) - x_k$, since k is k -unsaturated. This means that: $f_{x+(0,d_{-\hat{i}})}([n]) - f_{x+(0,d_{-\hat{i}})}([n] \setminus k) \geq f_{x+(0,d_{-k})}([n]) - [f_{x+(0,d_{-k})}([n]) - x_k] = x_k$. Therefore k is \hat{i} -unsaturated. Now, we can apply Lemma 4.15 that $f(S) + (x + d')(U) = f_{x+d'}(T) \leq f(T)$. Summing this with: $x([n] \setminus T) = f([n]) - f(T)$ we get: $x([n] \setminus T) + (x + d')(U) \leq f([n]) - f(S)$. To see this holds with equality, let x^f be the final allocation of the algorithm and notice that: $f(S) \geq x^f(S) = f([n]) - x^f([n] \setminus S) = f([n]) - [x([n] \setminus T) + (x + d')(U)]$. This shows us that:

$$x^f([n] \setminus S) = x([n] \setminus T) + (x + d')(U) = f([n]) - f(S)$$

which establishes that the invariant is preserved from one iteration to another. ■

References

- [1] G. Aggarwal, S. Muthukrishnan, D. Pál, and M. Pál. General auction mechanism for search advertising. In *WWW*, pages 241–250, 2009.
- [2] S. Alaei, K. Jain, and A. Malekian. Walrasian equilibrium for unit demand buyers with non-quasi-linear utilities. *CoRR*, abs/1006.4696, 2010.
- [3] L. M. Ausubel. An efficient ascending-bid auction for multiple objects. *American Economic Review*, 94, 1997.
- [4] B. Baisa. Auction design without quasilinear preferences. Working Paper, 2013.
- [5] J.-P. Benoit and V. Krishna. Multiple-object auctions with budget constrained bidders. *Review of Economic Studies*, 68(1):155–79, January 2001.
- [6] S. Bhattacharya, V. Conitzer, K. Munagala, and L. Xia. Incentive compatible budget elicitation in multi-unit auctions. In *SODA*, pages 554–572, 2010.

- [7] S. Bikhchandani, S. de Vries, J. Schummer, and R. V. Vohra. An ascending vickrey auction for selling bases of a matroid. *Operations Research*, 59(2):400–413, 2011.
- [8] C. Borgs, J. T. Chayes, N. Immorlica, M. Mahdian, and A. Saberi. Multi-unit auctions with budget-constrained bidders. In *ACM Conference on Electronic Commerce*, pages 44–51, 2005.
- [9] S. Chawla, D. L. Malec, and A. Malekian. Bayesian mechanism design for budget-constrained agents. In *ACM Conference on Electronic Commerce*, pages 253–262, 2011.
- [10] Y.-K. Che and I. Gale. Standard auctions with financially constrained bidders. *Review of Economic Studies*, 65(1):1–21, January 1998.
- [11] R. Colini-Baldeschi, M. Henzinger, S. Leonardi, and M. Starnberger. On multiple keyword sponsored search auctions with budgets. In *ICALP (2)*, pages 1–12, 2012.
- [12] N. R. Devanur, B. Q. Ha, and J. D. Hartline. Prior-free auctions for budgeted agents. In *EC*, pages 287–304, 2013.
- [13] S. Dobzinski, R. Lavi, and N. Nisan. Multi-unit auctions with budget limits. *Games and Economic Behavior*, 74(2):486–503, 2012.
- [14] S. Dobzinski and R. P. Leme. Efficiency guarantees in auctions with budgets. In *Proceedings of the 41st International Colloquium on Automata, Languages, and Programming*, ICALP’14, 2014.
- [15] P. Dütting, M. Henzinger, and I. Weber. An expressive mechanism for auctions on the web. In *Proceedings of the 20th international conference on World wide web*, WWW ’11, pages 127–136, New York, NY, USA, 2011. ACM.
- [16] A. Fiat, S. Leonardi, J. Saia, and P. Sankowski. Single valued combinatorial auctions with budgets. In *ACM Conference on Electronic Commerce*, pages 223–232, 2011.
- [17] G. Goel, V. S. Mirrokni, and R. Paes Leme. Polyhedral clinching auctions and the adwords polytope. In *STOC*, pages 107–122, 2012.
- [18] G. Goel, V. S. Mirrokni, and R. Paes Leme. Clinching auctions with online supply. In *SODA*, 2013.
- [19] J.-J. Laffont and J. Robert. Optimal auction with financially constrained buyers. *Economics Letters*, 52(2):181–186, August 1996.
- [20] A. Malakhov and R. V. Vohra. Optimal auctions for asymmetrically budget constrained bidders. Discussion Papers 1419, Northwestern University, Center for Mathematical Studies in Economics and Management Science, Dec. 2005.
- [21] E. S. Maskin. Auctions, development, and privatization: Efficient auctions with liquidity-constrained buyers. *European Economic Review*, 44(4-6):667–681, May 2000.
- [22] S. Morimoto and S. Serizawa. Strategy-proofness and efficiency with nonquasi-linear preferences: A characterization of minimum price walrasian rule. ISER Discussion Paper 0852, Institute of Social and Economic Research, Osaka University, Aug. 2012.
- [23] R. Myerson. Optimal auction design. *Mathematics of Operations Research*, 6(1):58–73, 1981.

- [24] M. M. Pai and R. Vohra. Optimal auctions with financially constrained bidders. Discussion papers, Northwestern University, Center for Mathematical Studies in Economics and Management Science, Aug 2008.
- [25] A. Schrijver. *Combinatorial Optimization - Polyhedra and Efficiency*. Springer, 2003.
- [26] E. Williams. Target cpa bidding: A new way to meet your roi goals with conversion optimizer. <http://adwords.blogspot.com.br/2010/05/target-cpa-bidding-new-way-to-meet-your.html>, 2010.

A Appendix: Missing Proofs

Proof of Lemma 2.1 : Let $\tilde{v}_1 \geq \tilde{v}_2 \geq \dots \geq \tilde{v}_n$. The outcome of VCG allocates all the units to player 1 and nothing to the other players, i.e., $x_1 = s$, $x_i = 0$ for $i \neq 1$. Also, $\pi_1 = \tilde{v}_2$ and $\pi_i = 0$ for $i \neq 1$.

Let (x', π') be a Pareto-improvement. Since the utility of 1 improves, it must be the case that: $v_1 x'_1 - \pi'_1 \geq v_1 s - \tilde{v}_2 s$. Therefore: $\pi'_1 \leq v_1 x'_1 - v_1 s + \tilde{v}_2 s$. Now, for each player $i \neq 1$, $\pi_i \leq \beta_i x_i \leq \tilde{v}_2 x_i$. So: $\sum_i \pi'_i \leq \tilde{v}_2 s + (\tilde{v}_2 - v_1)(s - x'_1)$. But since it is a Pareto-improvement, $\sum_i \pi'_i \geq \sum_i \pi_i = \tilde{v}_2 s$, so $x'_1 = s$. Which implies that $x = x'$, $\pi = \pi'$, which is a contradiction. ■

Note on the proof of Lemma 2.1: A alternative proof of Lemma 2.1 is to note that for multi-unit auctions and just average budget constraints, the polyhedral clinching auction (Algorithm 2) boils down to VCG on $\min\{v_i, \beta_i\}$. Beyond multi-unit auctions, however, this is not the case anymore.

Proof of Lemma 4.16 : This is trivially true in the first time the algorithm reaches point (♣), since at that point: $f([n]) = f_{x+d}([n]) = [f([n]) - f([n] \setminus i)] + f([n] \setminus i) = x_i + f([n] \setminus i) = f_{x+(0, d_{-i})}([n])$, after all, at that point $d_i = \infty$ for all i . Now, we will show this invariant is preserved the next time point (♣) is reached again.

Let d' be the demand vector after the price of \hat{i} increased. Then clinched amounts are: $\delta_i = f_{x+d'}([n]) - f_{x+(0, d'_{-i})}([n])$. The vector x is then updated to $x + \delta$ and d' to $d' - \delta$. We want to show that: $f_{(x+\delta)+(d'-\delta)}([n]) = f_{(x+\delta)+(0, d'_{-i}+\delta_{-i})}([n])$. Which is: $f_{x+d'}([n]) = f_{x+(\delta_i, d'_{-i})}([n])$.

By Lemma 4.14, $f_{x+(0, d'_{-i})}([n]) = \min\{f_{x+d'}([n]), f_{x+d'}([n] \setminus i) + x_i\}$. If the minimum is the first term, then $\delta_i = 0$ and we are done. If it is the second term, then: $\delta_i = f_{x+d'}([n]) - f_{x+d'}([n] \setminus i) - x_i$, in which case: $f_{x+(\delta_i, d'_{-i})}([n]) = \min\{f_{x+d'}([n]), f_{x+d'}([n] \setminus i) + x_i + \delta_i\} = f_{x+d'}([n])$. ■

Proof of Lemma 4.17 : See that this invariant is preserved through the execution of the algorithm. It is trivially true in the beginning since $d_i = \infty$ for all i . Now, if the invariant holds at point (♣), then by Lemma 4.16 $f([n]) = f_{x+d}([n]) = f_{x+(0, d_{-i})}([n])$. Now, when the price increases, the demands are updated and clinching happens, the value of $x + (0, d_{-\hat{i}})$ doesn't change, since $x_{\hat{i}}$ doesn't change and for $i \neq \hat{i}$, $x_i + d_i$ doesn't change, since x_i is updated to $x_i + \delta_i$ and d_i is updated to $d_i - \delta_i$. Therefore if \tilde{x}, \tilde{d} are the values of allocation and demands the next time point (♣) is reached, then: $f_{\tilde{x}+\tilde{d}}([n]) = f_{\tilde{x}+\tilde{d}_{-\hat{i}}}([n]) = f_{x+(0, d_{-\hat{i}})}([n]) = f([n])$. ■

Proof of Lemma 4.18 : Notice it is enough to show that $f_{x+(0, d_{-i})}([n]) = f_{x+(0, d_{-i})}([n] \setminus i) + x_i$ since one can extend for every set $S \ni i$ using submodularity, after all: $x_i \geq f_{x+(0, d_{-i})}(S) - f_{x+(0, d_{-i})}(S \setminus i) \geq f_{x+(0, d_{-i})}([n]) - f_{x+(0, d_{-i})}([n] \setminus i) = x_i$.

When point (\clubsuit) is reached for the first time in the algorithm, the invariant $f_{x+(0,d_{-i})}([n]) = f_{x+(0,d_{-i})}([n] \setminus i) + x_i$ holds trivially since $x_i = f([n]) - f([n] \setminus i) = f_{x+(0,d_{-i})}([n]) - f_{x+(0,d_{-i})}([n] \setminus i)$, by Lemma 4.17 and the fact that at this point $d_i = \infty$ for all i .

Now we will show that the invariant is preserved between two consecutive visits to point (\clubsuit) . Say that x, d is the allocation and demands in point (\clubsuit) just before we increase price p_i . Let d'_i be the new demand for i and $d' = (d'_i, d_{-i})$. We want to show that: $f_{x+(0,d'_{-i})}([n]) = f_{x+(0,d'_{-i})}([n] \setminus i) + x_i$. This is trivial for $i = \hat{i}$, since the expression is unaffected. For $i \neq \hat{i}$, we can use Lemma 4.14:

$$f_{x+(0,d'_{-i})}([n]) - f_{x+(0,d'_{-i})}([n] \setminus i) = \min\{f_{x+(0,d_{-i})}([n]), f_{x+(0,d_{-i})}([n] \setminus \hat{i}) + x_{\hat{i}} + d'_{\hat{i}}\} - \min\{f_{x+(0,d_{-i})}([n] \setminus i), f_{x+(0,d_{-i})}([n] \setminus \{i, \hat{i}\}) + x_{\hat{i}} + d'_{\hat{i}}\}$$

Now we can analyze four cases based on which expression achieves the minimum. One of those four cases is impossible, since by submodularity: $f_{x+(0,d_{-i})}([n] \setminus \{i, \hat{i}\}) - f_{x+(0,d_{-i})}([n] \setminus i) \geq f_{x+(0,d_{-i})}([n]) - f_{x+(0,d_{-i})}([n] \setminus \hat{i})$, the minimum can't be achieved by $f_{x+(0,d_{-i})}([n])$ in the first and by $f_{x+(0,d_{-i})}([n] \setminus \hat{i}) + x_{\hat{i}} + d'_{\hat{i}}$ in the second. Now, we proceed by analyzing the remaining cases:

- first / first: $f_{x+(0,d'_{-i})}([n]) - f_{x+(0,d'_{-i})}([n] \setminus i) = f_{x+(0,d_{-i})}([n]) - f_{x+(0,d_{-i})}([n] \setminus i) = x_i$ by the invariant.
- second / second: $f_{x+(0,d'_{-i})}([n]) - f_{x+(0,d'_{-i})}([n] \setminus i) = [f_{x+(0,d_{-i})}([n] \setminus \hat{i}) + x_{\hat{i}} + d'_{\hat{i}}] - [f_{x+(0,d_{-i})}([n] \setminus \hat{i}) + x_{\hat{i}} + d'_{\hat{i}}] \geq f_{x+(0,d_{-i})}([n]) - f_{x+(0,d_{-i})}([n] \setminus i) = x_i$. The inequality in the other direction is trivial.
- second / first: in this case by the fact that the minimum is achieved by the first term in the second expression, $f_{x+(0,d_{-i})}([n] \setminus i) \leq f_{x+(0,d_{-i})}([n] \setminus \{i, \hat{i}\}) + x_{\hat{i}} + d'_{\hat{i}}$, therefore: $f_{x+(0,d'_{-i})}([n]) - f_{x+(0,d'_{-i})}([n] \setminus i) = f_{x+(0,d_{-i})}([n] \setminus \hat{i}) + x_{\hat{i}} + d'_{\hat{i}} - f_{x+(0,d_{-i})}([n] \setminus i) \geq f_{x+(0,d_{-i})}([n] \setminus \hat{i}) - f_{x+(0,d_{-i})}([n] \setminus \{i, \hat{i}\}) \geq x_i$. The inequality in the other direction is trivial.

This shows that the invariant holds even after we decrease the d_i . Before we reach point (\clubsuit) again, allocation and demands change because of clinching. This, however, doesn't change the invariant, since clinching adds an extra δ_i amount to x_i but subtracts the same amount from d'_i , so $x + d$ is constant. Therefore, clinching doesn't affect the invariant. ■